COURSE OF STUDY IN SECONDARY MATHEMATICS IN THE UNIVERSITY HIGH SCHOOL, THE UNIVERSITY OF CHICAGO

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INTRODUCTION

In the following course of study it is aimed so to reorganize the body of secondary-school mathematics as to fit the subject effectively to the needs of the students and to make it more productive for mental life and growth. Since 1903 the departments of mathematics of the University High School and the School of Education have been developing a solution of the problem of mathematical reorganization along the lines of correlation. By the bringing together of subjects that are closely related, each is reinforced by the aid of the others, and the entire work in these subjects is unified and vitalized. For example, algebra and geometry supplement each other. Both are used to express facts about quantity. The formula and the graph are only different ways of expressing the law of a group of numerical facts. Each states the facts in generalized form and thus makes the deduction of any number of particular cases possible. Moreover, when the two forms of thought are correlated in a single course of instruction the student's comprehension of quantity is at the same time deepened and simplified—deepened because of the more enduring impression made upon the mind; simplified because the double method of attack makes it easier to overcome difficulties by supplying always a strategic alternative.

The student will see the advantages of having various modes of treating the facts of quantity. Thus, he is made to realize the value of algebra by seeing the superiority of algebraic methods in

important respects as compared with arithmetic and geometric methods.

When the various mathematical subjects are treated separately, each tends to take on the rigid form of the final science. This leads inevitably to a certain formalism in the mode of presentation. Such formalism is not the best method for the high-school pupil. Correlation helps to avoid excessive formalism. Rigor is not carried beyond the understanding of the pupil.

Leading mathematicians and professors of the teaching of mathematics have long recognized the relationship between the various subjects of secondary mathematics, especially algebra and geometry, and have advocated the desirability of teaching them together. The details of a plan for this correlation are being worked out in the University High School and the content of the first-year course is now well organized and is given in *First-Year Mathematics*, a textbook published by the University of Chicago Press. The second-year course is contained in *Second-Year Mathematics*, also published by the University of Chicago Press. Work on the third- and fourth-year courses is progressing and it is hoped to publish them in the near future.

AIMS AND VALUES

Among the purposes of secondary mathematics the following are emphasized:

1. A knowledge of the fundamental facts and principles of mathematics, needed by students because of the correlation of these facts and principles with other studies, such as science, drawing, geography, and arithmetic (verification of the familiar geometric formulas used in arithmetic), and because of their usefulness in science (astronomy, physics, chemistry, geology, economics, all use mathematics), engineering, designing, architecture, navigation, railroad building. The best interest in mathematics cannot be secured without a considerable number of practical problems that come within the comprehension and observation of the ordinary pupil.

2. Development of spatial and pictorial imagination; i.e., the ability to visualize objects, relations, and conditions. This train-

ing, which cannot be secured from other high-school subjects, is a real need in life, especially for those who cannot go beyond the high school.

3. Development of functional intuition; i.e., of the appreciation of the dependence of one magnitude upon another. The pupil learns to attack new problems with the fertile question: "What relations are here involved?" rather than the sterile: "What is the unknown here?" The existence of this dependence falls within the experience of every pupil.

4. The disciplinary value; e.g., training in accurate thought and power of concentration; the acquisition of orderly habits of precise oral and neatly written expression and, not at all the least, of honesty, i.e., of saying exactly what one means and meaning exactly what one says; training in setting out to do a specific thing and doing precisely that thing; i.e., training in the habit of "making good." The disciplinary value of mathematics always has been considered one of the principal values of mathematics.

5. Improving the study-habits of the student. Since supervised study is easily introduced in mathematics, this subject is especially valuable as a means of teaching students how to study.

FIRST-YEAR MATHEMATICS

GENERAL STATEMENT

In his elementary mathematics the student has learned to add, subtract, multiply, and divide numbers and to use his knowledge of these operations in the solution of problems. He has become acquainted with the common geometrical forms and many relations of geometrical magnitude. In many respects algebra is like arithmetic. It consists of the study of the operations and is an effective tool for solving problems. The study of algebra enables the student to get a better understanding of the principles underlying arithmetic, to express laws in brief form, and to abbreviate the language in the solution of problems. Hence, in a first course in secondary mathematics the student should continue the study of numbers and extend his knowledge of the fundamental notions of geometry.

In planning the work of the year, the following facts have been kept in mind:

Each of the various divisions of secondary mathematics—arithmetic, algebra, geometry, and trigonometry—includes simple principles relatively easy to master and also difficult complex principles. The simple principles are best suited to beginners, and may therefore be brought together in an introductory course which leads up to the more complex aspects of these various branches of mathematical science.

When they make the acquaintance of only one subject in mathematics during the first year, many students fail to get an insight into secondary mathematics. Discouraged by failure in one subject, these pupils do not continue the study, and thus they miss the opportunity to discover that they can be successful in another subject. However, in an introductory course in which algebra and geometry are taught together, success in one field will arouse an interest and enthusiasm which will encourage the student to attack the other with increased vigor. The result will be a gain in mathematical power and no less in general training. In the first-year course, geometry is used throughout to illustrate algebraic processes, while algebra carries on the reasoning in the compact and abstract symbols which generalize quantitative facts to an extent which is impossible in graphic expression.

The fundamental notions of trigonometry, which are commonly kept from the student until the third or fourth year of high school, appeal to him because of their usefulness as tools in problem-solving. Hence these notions are introduced at an early stage and they are so presented as to create no material difficulty for the beginner.

AIMS

Ability to solve simple equations in one or more unknowns, to solve quadratic equations in one unknown, to maintain algebraic expressions and formulas, and to represent given data in algebraic symbols.

To review, clarify, and develop informally the fundamental notions of geometry such as point, line, plane, surface, solid, angle, parallel lines, etc., and certain fundamental facts of geometry, as equality of angles, of line-segments, congruence, similarity, and symmetry of figures. Review of the formulas of geometry generally taught in arithmetic.

Ability to represent numerical facts graphically.

To acquire general familiarity with the instruments used in geometry.

To remove the great difficulties found by students when taking up logical geometry, by passing gradually and almost imperceptibly to the deductive method of demonstrative geometry.

METHODS

Algebra is introduced as a natural means of expressing facts about number, gradually becoming a symbolic language, especially adapted to stating conditions of a problem in a natural and helpful way. The growing difficulty and complexity of problems then lead to the necessity of learning how to manipulate algebraic symbols and expressions, and how to solve Algebraic equations. The symbolism of algebra thus becomes a highly clarifying instrument of problem-analysis and problem-solving.

Strong emphasis is placed upon intuition, observation, description, and motor control. At first students are taught to measure, to construct with rules and compass, to recognize the fundamental forms of geometry in the classroom and elsewhere. This suggests the geometric truths. Concepts and facts discussed are first expressed in terms already familiar to the student, then gradually translated into the precise language of geometry. The fact that measurements are at best only approximate, and are very tedious and difficult if carefully done, gradually leads to a desire for better methods and to a need for the logical method. In short, these experiences motivate the logical procedure. However, attempts at lengthy formal demonstration are not made. The method of proof is always informal, its aim being to establish geometric facts and to prepare for, not to develop skill in, logical demonstration.

The laws of algebra are carefully illustrated, and thus the student is enabled to avoid the danger of symbol-juggling without insight into the real meaning. Certain processes which belong together logically are separated in treatment because they present real difficulties for the beginner. Hence, wherever the processes are not needed as instruments of instruction they are taught separately; e.g., the meaning of positive and negative numbers, the

laws of signs, and the operations with positive and negative numbers are not studied until the pupil has become thoroughly familiar with unsigned literal numbers and with operations and laws of such literal numbers.

CONTENT

I. THE STRAIGHT LINE.

Measurement of line-segments.
Ways of expressing facts about quantity.

II. ADDITION AND SUBTRACTION.

Graphical addition and subtraction.
Perimeters.
Algebraic addition and subtraction.

III. THE EQUATION.

Use of axioms in solving equations.
Problems to be solved by the aid of the equation.

IV. ANGLES.

Classification of angles.
The measurement of angles.
The use of the protractor in measuring angles.
The sum of the angles of a triangle.
The sum of the exterior angles of a triangle.
To draw an angle equal to a given angle.

V. AREAS AND VOLUMES. MULTIPLICATION.

Area of a square.
Area of a rectangle.
Volume of cube and parallelopiped.
Graphing equations.
Multiplication of monomials.
Addition of monomials.
Multiplication of a polynomial by a monomial.
Multiplying polynomials by polynomials.
Area of parallelogram and triangle.

VI. ANGLE-PAIRS.

Adjacent angles.
The sum of the adjacent angles about a point on one side of a straight line.
The sum of the angles at a point.
Supplementary angles.
Complementary angles.

Opposite angles.
The acute angles of a right triangle.
Angle-pairs formed by two lines intersected by a third.

VII. PARALLEL LINES. LINES AND PLANES IN SPACE.

Parallel lines.
Angles of the parallelogram and trapezoid.
Models of geometrical solids.

VIII. MEASUREMENT OF LINES IN SPACE. SIMILAR FIGURES.

Drawing to scale.
Ratio.
Similar figures.
Problems in similar figures.

IX. RATIO. VARIATION. PROPORTION.

Trigonometric rates.
Ratio.
Direct variation.
Inverse variation.
Proportion.
Proportionality of areas.

X. CONGRUENCE OF TRIANGLES.

Congruence.
The isosceles and the equilateral triangle.
The right triangle.

XI. CONSTRUCTIONS. SYMMETRY. CIRCLE.

The fundamental constructions summarized and proved.
Applications of the fundamental constructions.
Symmetry.
The circle.

XII. POSITIVE AND NEGATIVE NUMBERS. THE LAWS OF SIGNS.

Uses of positive and negative numbers.
Graphing data.
Addition of positive and negative numbers.
Subtraction of positive and negative numbers.
Law of signs in multiplication.
Multiplication by zero.
Product of several factors.
Law of signs for division.

XIII. ADDITION AND SUBTRACTION.

Review of the laws of addition.
Addition of monomials.

Addition of polynomials.
Subtraction of monomials.
Subtraction of polynomials.
Removal of parenthesis.

XIV. Multiplication and Division.

Multiplication of monomials.
Multiplication of polynomials by monomials.
Multiplication of polynomials by polynomials.
Multiplication of arithmetical numbers.
Division of monomials.

XV. Special Products. Factoring. Quadratic Equations.

The square of a binomial.
Factoring trinomial squares.
Product of the sum of two numbers by their difference.
Factoring the difference of two squares.
The product of two binomials of the form $(ax+b)(cx+d)$.
Factoring trinomials of the form ax^2+bx+c.
The theorem of Pythagoras.
Square root of arithmetical numbers.
Quadratic equations.
Quadratic equations solved by factoring.
Quadratic equations solved by completing the square.

XVI. Problems Leading to Equations of the First Degree in One Unknown.

Solution of problems and equations.
Geometric problems.
Problems involving number relations.
Motion problems.
Clock problems.
Problems on percentage and interest.
Mixture problems.
Lever problems.

XVII. Linear Equations Containing Two or More Unknown Numbers.

A system of two linear equations.
Graphical method of solving a system of equations.
Algebraic solution of equations in two unknowns.
Geometric problems.
Motion problems.
Miscellaneous problems.
Fractional equations.
Systems of three or more linear equations.

XVIII. The Formula.

The formula as a general rule.
Evaluation of formulas.
Expressing one of the letters of a formula in terms of the others.

XIX. Review and Supplementary Questions and Problems.

TEXTBOOK

Breslich's *First-Year Mathematics*, published by the University of Chicago Press, is used in the first-year course.

TIME REQUIRED OF STUDENTS

Five one-hour class periods a week throughout the school year are given to mathematics. Probably an average of fifteen minutes of the period is used for supervised study where special attention is given to the individual differences existing among students. An average of about fifteen minutes a day is expected for home work. This work is generally of the same type as the work done in the classroom.

STANDARDS OF ATTAINMENT

The student who expects to take only one year of mathematics in the high school has acquired a knowledge of such geometric facts as he is most likely to make use of in later life; he knows enough algebra to prefer the algebraic method of solving problems to the less effective arithmetic method; and he has had a good review of the fundamental processes with arithmetic numbers.

The student who expects to go on with the study of mathematics has laid a good foundation upon which to build the future work.

TYPE LESSONS

I. *A lesson illustrating how to take up the study of the addition axioms.*—Line-segments are used to establish a concrete basis for this study.

The teacher draws on the blackboard two pairs of equal line-segments as AB and CD, and EF and GH, Fig. 1 (see p. 657). A student is then asked to draw on the board the sums of $AB+EF$ and $CD+GH$. Another student measures these sums and compares them. He finds that they are equal.

The problem is then repeated in a somewhat different form: Letting a, b, c, and d be the lengths of four segments such that $a=b$, and $c=d$, show by measuring that $a+c=b+d$. By means of these problems the student is led to state the addition axiom in his own

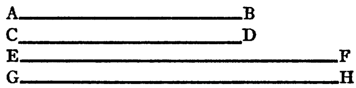

FIG. 1

words. He finds in the textbook, on the other hand, a statement of the axiom with which to compare his own: Equals added to equals give equals.[1] The second problem makes him familiar, while he is learning the statement, with the form in which this axiom is usually applied. At the same time he is acquiring valuable training in the ability to measure—so important for all graphical work.

II. *A lesson illustrating how abstract discussions are introduced with concrete illustrations.*—The use of axioms in solving equations is preceded by the solving of equations by the aid of the balance.

In making a study of the equation we must begin with some very simple problems in order that we may clearly understand the new laws to be developed. If these laws are mastered in connection with simple cases, it will be easy to apply them later to more complicated and difficult cases. Let us solve the following problem: A bag of grain of unknown weight, w ounces, together with an 8-oz. weight, just balances an 18-oz. weight. How much does the bag of grain weigh?

The problem may be stated in an equation, thus:

$$w+8=18. \quad \text{Find } w.$$

Suppose 8 oz. to be taken from each pan, Fig. 2, giving

$$w=10$$

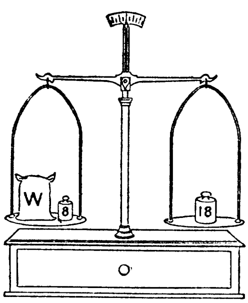

FIG. 2

The bag of grain weighs 10 oz. Showing $w+8=18$.

At this point it is necessary to define two mathematical terms: An equation, as $w+8=18$, may be regarded as an expression of

[1] *First-Year Mathematics*, p. 17.

balance between the numbers on the two sides of the equality sign. The number to the left of the equality sign is the left side, or *left member* of the equation, the number to the right is the right side, or *right member.*

Thus, in the equation $a+5=7$, $a+5$ is the left member and 7 is the right member.

Several equations are now written on the blackboard, and pupils state in words problems expressed by these equations.

These equations are then solved by the aid of the balance.

We shall now learn how to solve an equation without the use of the balance. An equation, as $w+8=18$, may be regarded as stating the question: What number added to 8 gives 18? It has been shown that the answer may be found by interpreting the equation as a problem in weighing and then taking 8 oz. from both pans of the balance. Just as the scales will balance if the same number of weights are taken from each pan, we may subtract the same number from both sides of an equation and get another equation. The work of finding the unknown number this way may be arranged in written form thus:

$$\begin{array}{rr} \text{Let } w+8 = & 18 \\ 8 = & 8 \\ \hline \text{Then } w \quad = & 10 \end{array}$$

For, if the same number be subtracted from equal numbers, the remainders are equal (subtraction axiom).

To test the correctness of the result, replace the unknown number in the original equation by 10, obtaining $10+8=18$. Since both members of the equation reduce to the same number, the result $w=10$ is correct.

This is followed by solving a number of equations of the same type as above using the subtraction axiom.

III. *This lesson illustrates the "experimental method" in geometry.*—We are to show that the sum of the angles of a triangle is 180°. At first the truth of the theorem is shown by intuition as follows:

Draw a triangle. Cut the triangle from the paper. Tear off the corners and place the angles adjacent to each other, Fig. 3.

What seems to be the sum of the angles of the triangle? The student states that the sum is a straight angle or 180°.

The same fact is now shown in a different way. That the sum of the angles of a triangle is 180° can be shown by rotating a stick

FIG. 3

or a pencil successively through the angles, as follows: Draw a triangle. Place a pencil or stick in position 1, Fig. 4, and note the direction in which it is pointing. Rotate the pencil through angle x. Then move it along AB to position 2. Turn the pencil through angle y and move it along BC to position 3. Turn it through angle z to position 4. The pencil has now rotated through an amount equal to $x+y+z$. Note the direction in which the pencil is pointing in the last position. Through what part of a complete turn has it rotated? Through how many right angles? Through how many degrees?

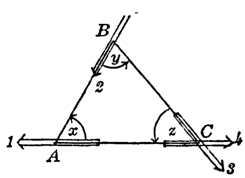

FIG. 4

The student is now asked to express the new fact in algebraic symbols: State by an equation the number of degrees in the sum of the angles x, y, and z of a triangle.

By means of this equation it is possible to find an angle of a triangle when the other two are known. This is of great importance, e.g., it enables surveyors to find all angles of a triangle although they may be able to measure directly only two angles.

The new principle is now used in the solution of a number of problems. In solving the next problems observe the following steps: Make a sketch of the triangle, denoting the angles as given in the problem. To obtain the equation use the theorem that the sum of the angles of a triangle is 180°. Solve the equation and find the values of the angles.

Illustrative problem: The difference between two angles of a triangle is 20°, and the third angle is 36°. Find the unknown angles.

Let x be one of the angles, Fig. 5.

Then $x+20$ is the second angle.

The third angle is given equal to 36.

Therefore $x+x+20+36=180$, since the sum of the angles of any triangle is 180°.

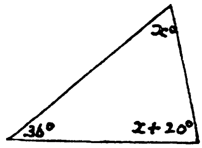

FIG. 5

Solving for x, we have

$$\begin{aligned} x &= 62 \\ x+20 &= 82 \\ 36 &= 36 \\ \hline x+x+20+36 &= 180 \end{aligned}$$

Check:

IV. *The following lesson illustrates the "informal method of proof," used in the later part of the course, when a change is made from the experimental to the logical procedure.*—It is proposed to prove the following theorem: The shortest distance from a point to a line is the perpendicular from the point to the line.

No attempt is made to observe the form used in demonstrative geometry, i.e., to state first the hypothesis, then the conclusion, and finally the proof. Instead, the student begins at once to reason as follows: Let AB be perpendicular to DE, Fig. 6, and let AC be any other line from A to DE.

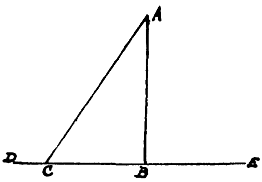

FIG. 6

Angle C in triangle ABC is acute, for if one angle of a triangle is 90° the other angles are less than 90°.

Therefore angle C must be less than B, which is a right angle.

It follows that AB is less than AC, for if two angles of a triangle are unequal, the sides opposite are unequal, the greater side being opposite the greater angle.

V. *The following lesson shows how geometry is used to establish a concrete basis for abstract discussions.*—The law of signs in multiplication is taken up after the student has had a half-year's work in high-school mathematics. It is approached by means of four problems, as follows:

1. Find the product of (+4) by (+3).

Solution: Since $(+3)(+4)$ is the same as $(3)(+4)$, it follows that $(+3)(+4)$ equals $(+4)+(+4)+(+4)=(+12)$. Geometrically this means that to multiply $(+4)$ by $(+3)$ is to lay off $(+4)$ three times in its own direction, Fig. 7.

Thus $$(+3)(+4)=(+12).$$

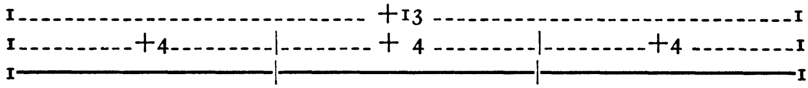

FIG. 7

2. Find the product of (-4) by $(+3)$.

Solution: Since $(+3)(-4)$ is the same as $(3)(-4)$, $(+3)(-4)=(-4)+(-4)+(-4)=(-12)$.

Make a drawing for $(+3)(-4)$, i.e., lay off (-4) three times in its own direction.

Thus, $$(+3)(-4)=(-12)$$

3. Find the product of $(+4)$ by (-3).

Solution: Assuming that the commutative law holds for positive and negative numbers (the student is familiar with this law) then it follows that

$$(-3)(+4)=(+4)(-3)=(-12)$$

Notice that the same result is obtained by laying off $(+4)$ three times in the direction opposite to its own direction. Make a drawing for this product.

4. Find the product of (-4) by (-3).

According to problem 3, this means that (-4) is to be laid off three times in the direction opposite to that of (-4).

Thus, $$(-3)(-4)=(+12)$$

Then, from a study of the results of problems 1–4, the law of signs is deduced.

SECOND-YEAR MATHEMATICS

ORGANIZING PRINCIPLES

The combined type of material of the mathematics of the first year is to be carried forward through the second year, the emphasis being on geometry.

The operations and laws of arithmetic should be reviewed wherever opportunity is offered and occasion warrants, as in the evolution

of formulas, in the introduction of new algebraic topics, and in problems of computation.

The algebraic ground gained in the first year should be held and extended at least as far as is customary with the algebra before the third year.

The geometry should be largely of the demonstrative type.

The student should receive training in both plane and solid geometry. Many theorems of solid geometry closely related to corresponding theorems in plane geometry should be proved in the second year, thus training the student in both two- and three-dimensional thinking.

The study of trigonometry begun in the first year is to be continued. Trigonometric methods should often replace algebraic and geometric methods, giving the student the opportunity to see some of the advantages of trigonometry over algebra and geometry.

AIMS AND VALUES

To complete the study of plane geometry and algebra begun in the first year. By this is meant to cover the essentials of what is commonly required of students in the first two years of secondary schools in this country.

To include in addition the elementary notions of trigonometry; the application of three trigonometric functions (sine, cosine, and tangent) to the solution of the right triangle, and to a number of practical problems; the development of some of the fundamental relations between these functions, the use of these functions in proving geometric theorems.

To begin and to make a real advance in the study of solid geometry before completing plane geometry. This includes the theorems on lines and planes in space, diedral angles, areas of surfaces, and volumes of solids, omitting the discussions of polyedral angles and spherical polygons.

No topical treatment of the theory of limits is intended. Such a treatment is not believed to belong to the early years of the high-school course. However, the question of the existence of incommensurable lines and numbers is raised, examples of these are given, and the notion of the limit of a sequence is developed.

The material as arranged in this course opens to the student a broader, richer, a more useful and therefore more alluring, field of ideas and lays a more stable foundation for future work than does any separate treatment. A great saving of the student's time is effected, by developing arithmetic, algebra, geometry, and trigonometry side by side. This also makes unnecessary the long and tiresome reviews usually given at the beginning of each subject, replacing them by frequent incidental reviews leading immediately to an extension of the subject.

Often a high-school pupil fails rightly to esteem a high-school subject because he cannot discern its bearing on what has preceded and on what is to follow. But having seen the closeness of the relation between the subjects, he does not lose view of the familiar fields while at the same time he obtains an outlook into neighboring and more remote fields. There is, thus, the economy resulting from accomplishing more work in less time and from the performance of tasks that are intelligently motivated.

METHODS

In the first-year course the student has gained a thorough understanding of the fundamental notions of geometry. Hence in the second year somewhat formal methods are introduced from the start. Even before this, the advantage of the reasoning process over the process of measuring has been recognized. Mathematical fallacies and optical deceptions are now used to make the need of a logical proof still more apparent.

To develop in the pupil a sort of geometrical strategy, i.e., to attack, to take possession, and to exploit a geometrical problem, methods of proof are discussed and emphasized, not once for all, but throughout the course. To cultivate versatility and system, students are taught to choose between various methods of proof and to follow some definite plan rather than to trust the chance of stumbling upon a proof. Many model proofs are therefore given, while in other proofs statements or reasons that should be apparent to the student are omitted with the purpose of making him independent and of developing his powers of argumentation.

The old method of dividing the subject of geometry into a few books has been abandoned as being of only traditional value. The

course is divided into a number of short chapters, each dealing with one or a few central topics. This arrangement is far better adapted to study by high-school students, since the aims and purposes of the several chapters are easily seen. It is more economical of the student's time and energy than the old method.

Since the usefulness of a study is what always appeals most strongly to a beginner, this phase is emphasized throughout the course. The importance and the significance of geometrical facts in the affairs of everyday life are impressed upon the pupil.

The plan of introducing definitions whenever needed and not before, which is used in the first-year course, has been followed also in the second year.

By the employment of algebraic notation and by the continued application of the equation to geometrical matters, a firm hold is kept on algebra. New algebraic topics are developed when opportunity and need arise. Thus, elimination by comparison and by substitution, so frequently needed in proofs and in the solution of exercises, is taught very early. The solution of the quadratic equation by means of the formula, the operations with fractions, and factoring are all reviews or further extensions of topics whose study was begun in the first year.

A number of geometric proofs have been simplified by the use of trigonometry.

CONTENT

I. Methods of Proof.
  Logic, Geometrical fallacies.
  Methods of proofs.

II. Methods of Elimination.
  Problems leading to equations in two unknowns. Elimination by addition, subtraction, substitution, comparison.

III. Quadrilaterals.
  Parallelogram.
  Trapezoid.
  Kite. Symmetry. Loci.
  Primatic surfaces.
  Lines and planes in space. Diedral angles.

IV. Proportional Line-Segments.
  Uses of proportional line-segments. Proportional segments. Constructions.
  Lines and planes in space.

V. PROPORTION. FACTORING. VARIATION.

Fundamental theorems.
Factoring.
Processes of obtaining proportions from given proportions.
Relation between proportion and variation.

VI. SIMILAR POLYGONS.

Uses of similar triangles.
Theorems on similar figures.

VII. RELATIONS BETWEEN THE SIDES OF TRIANGLES. THEOREM OF PYTHAGORAS AND ITS GENERALIZATIONS. QUADRATIC EQUATIONS.

Similarity of the right triangle.
Relations of the sides of a right triangle.
Quadratic equations.
Generalizations of the theorem of Pythagoras.

VIII. TRIGONOMETRIC RATIOS. QUADRATIC EQUATIONS IN TWO UNKNOWNS.

Determination of the values of trigonometric ratios of a given angle.
Determination of the values of the functions when the value of one function is given.
Exact values of the functions of 30°, 45°, and 60°.
Applications of the trigonometric functions.
Relations of trigonometric functions.

IX. THE OPERATIONS WITH FRACTIONS. FRACTIONAL EQUATIONS.

Addition and subtraction of fractions.
Multiplication of fractions.
Division of fractions.
Complex fractions. Fractional equations.
Trigonometric relations.

X. THE CIRCLE. THE SPHERE.

Review and extension of the properties of circles.
Diameters, chords, and arcs.
Tangent circles.

XI. MEASUREMENT OF ANGLES BY ARCS OF THE CIRCLES.

Measurement of angles.
Problems of construction.
Miscellaneous exercises.

XII. PROPORTIONAL LINE-SEGMENTS IN CIRCLES.

XIII. INEQUALITIES.

Review and extension.
Lines and planes in space.

XIV. LINES AND PLANES IN SPACE.

XV. LOCI. CONCURRENT LINES.

XVI. REGULAR POLYGONS AND CIRCLES.
Length of the circle.

XVII. AREAS.
Comparison of areas.
Literal equations.
Area of the triangle.
Factoring.

XVIII. AREAS. AREAS OF POLYGONS. AREA OF THE CIRCLE.
Proportionality of areas.
Problem of construction.

TIME REQUIRED OF STUDENTS

Five one-hour class periods a week throughout the school year are given to second-year mathematics. An average of about fifteen minutes of the period is used for supervised study. An average of about thirty minutes a day is expected for home work.

TEXTBOOK

Breslich's *Second-Year Mathematics*, published by the University of Chicago Press, is used in the second-year course.

STANDARDS OF ATTAINMENT

While in the preceding course geometric facts were established mainly by the inductive and the experimental methods, the student now has learned to demonstrate the proofs of geometric facts and to attack the solution of problems and exercises by various methods. He has completed the work that is usually given in a course in plane geometry.

In algebra he has had a review and extension of his knowledge of the solution of equations by various methods, of factoring, of fractions, and of quadratic equations. He has covered what is ordinarily given in a one-year course of algebra and some of the work is given in "intermediate algebra."

He has made a beginning of the study of trigonometry. A large part of the solid geometry has been completed.

TYPE LESSON

The following lesson illustrates the method of presenting a theorem in geometry in the second-year course.—It is required to prove that if the alternate interior angles, formed by two parallel lines and a transversal, are equal, the lines are parallel.

Several students are asked to state the theorem. This is kept up until it becomes apparent that every member of the class has memorized the statement.

The class is then asked to think about the figure. A few moments later a student is called upon to draw the figure on the blackboard.

Then the hypothesis is stated definitely with reference to the figure, using the best symbolic notation.

Similarly the conclusion is obtained.

Even the poorest student in the class must be able to do the work up to this point.

Then follows the preliminary discussion (analysis).

Q. What are we to prove? A. $AB \parallel CD$.

Q. When are two lines parallel? A number of theorems are suggested by the class and the following is selected as the most promising for our purpose: Two lines are parallel, if they are perpendicular to the same line. This suggests drawing a helping line perpendicular to AB to prove that it is perpendicular to CD, as HG.

Q. When are two lines perpendicular? A. If they are at right angles.

Q. How may we prove angle G equal to a right angle? A. By proving $\angle G = \angle H$.

Q. How may we prove two angles equal? From a number of answers the congruent triangle method is selected.

Q. Who can prove triangles HEM and FGM congruent? The proof is given by a student.

We now retrace the order of the steps in this analysis and obtain the proof. After the proof is given, the essential steps are emphasized and the theorem is assigned for review as home work.

THIRD-YEAR MATHEMATICS

GENERAL STATEMENT

The work of the third year is a continuation of the study of the topics given in the preceding years, the emphasis being mainly on algebra and trigonometry.

AIMS

During the second year the study of plane geometry has been completed.

During the third year the study of algebra and trigonometry is to be continued. The work in solid geometry, begun in the first year and continued in the second, is to be completed in the third.

CONTENT AND METHODS

A. Equations

I. LINEAR EQUATIONS IN ONE UNKNOWN.

This work begins with verbal problems of a type more difficult than those of the first and second years. As need arises, the processes of addition and subtraction, multiplication and division of polynomials are reviewed. This is done by solving equations which are of the first degree when put in the simplest form.

II. LINEAR EQUATIONS IN TWO OR MORE UNKNOWNS.

Methods of solution studied in the preceding courses are reviewed and summarized. The solution by determinants is added. At the end of this work the student must be able to solve systems containing very complicated literal and fractional equations. Equivalent and inconsistent equations are studied.

III. QUADRATIC EQUATIONS IN ONE UNKNOWN.

Review of methods of solution: Graphical method, factoring method, methods of elimination, and the formula. At the end of this work the student must be able to solve very complicated quadratic equations. Many verbal problems are solved.

Incidentally there are reviews and extensions of the processes of factoring, extracting roots of polynomials, and of simplifying monomials containing radicals.

Discussion of the nature of the roots of a quadratic equation.

IV. QUADRATIC EQUATIONS IN TWO UNKNOWNS.

Graphing of quadratic equations makes the student familiar with the following lines: parabola, ellipse, hyperbola, circle, and a pair of

intersecting straight lines. The graphs illustrate the meaning of a solution, the number of solutions to be expected, and the process of the algebraic solution.

As in factoring, the student learns that he cannot solve *every* system of the second degree in two unknowns and proceeds to study *only certain* type forms. These forms suggest the method of solution. Some of these methods are applied even to systems of higher degree than the second.

In connection with the computations arising in problems the use of tables, slide rule, and graphs is taught.

V. EQUATIONS OF HIGHER DEGREE THAN THE SECOND.

The factoring method and the method of changing the equation into the form of a quadratic are used to solve the equations in this chapter. The student learns to form equations whose roots are given. Many verbal problems are solved.

VI. IRRATIONAL EQUATIONS.

The equations are solved either by squaring both sides, or by throwing them into the form of quadratic equations. Applications to solid geometry.

B. *Functions*

I. SERIES.

The binomial theorem and the formula for the *n*th term are developed and applied. Arithmetic and geometric progressions are studied.

II. FRACTIONS. The following topics are included:

The operations with fractions; complex fractions; fractional equations; problems leading to fractional equations.

III. VARIATION.

IV. EXPONENTS AND RADICALS.

C. *Trigonometry*

I. REVIEW AND EXTENSION: TRIGONOMETRIC RATIOS OF ACUTE ANGLES.

1. Definitions of six trigonometric ratios.
2. Constant value of any ratio for the same angle.
3. Equality of angles having equal trigonometric ratios.
4. Given the value of trigonometric ratio of an acute angle, to construct the angle and to obtain the values of the other trigonometric ratios.
5. Approximating by measurement the values of the trigonometric ratios of a given angle.
6. Changes of the trigonometric ratios of the angle A, as A increases from 0 to 90.

7. Trigonometric ratios of complementary angles.
8. Simple trigonometric equations.
9. Exact values of the trigonometric ratios of 30°, 45°, and 60°.
10. Solution of right triangles by means of natural functions.
11. Verbal problems.

II. TRIGONOMETRIC RATIOS OF POSITIVE AND NEGATIVE ANGLES OF ANY SIZE.

1. Necessary definitions.
2. Laws of the quality of the trigonometric ratios for the various quadrants.
3. Six fundamental relations between the trigonometric ratios of an angle.
4. Problems including all of the above.
5. Proofs of identities.
6. Changes of the trigonometric ratios of an angle A, as A increases from 0° to 360°.
7. The trigonometric ratios of $(-A)$ in terms of the ratios of A.
8. The quadrantal formulas.
9. Applications.

III. TRIGONOMETRIC RATIOS OF TWO ANGLES.

1. Addition and subtraction formulas.

   (a) $\sin (A+B)$; $\cos (A+B)$; $\tan (A+B)$; $\cot (A+B)$.
   (b) $\sin (A-B)$; $\cos (A-B)$; $\tan (A-B)$; $\cot (A-B)$.

2. Trigonometric ratios of twice an angle in terms of the ratios of the angle.

$$\sin 2A;\ \cos 2A;\ \tan 2A;\ \cot 2A.$$

3. Trigonometric ratios of half an angle in terms of the cosine of the angle:

$$\sin \frac{A}{2};\ \cos \frac{A}{2};\ \tan \frac{A}{2};\ \cot \frac{A}{2}.$$

4. Sum and difference of sines and cosines expressed as products.

IV. SOLUTION OF RIGHT TRIANGLES WITH LOGARITHMS.

1. Formal problems.
2. Verbal problems.
3. Isosceles triangles.
4. Regular polygons.

V. SOLUTION OF TRIANGLES IN GENERAL.

1. Law of sines.
2. Law of cosines.
3. Law of tangents.

4. Formulas for the case when three sides are given.
5. Discussion of the different cases and of the laws that apply.
6. Formal and verbal applications.
7. Area of triangle.
8. Circumscribed and inscribed circle.

VI. Radian Measure, General Values of an Angle, Trigonometric Equations, Inverse Functions.

1. Definition of radian.
2. Radian measure of angles. Problems involving radian measure.
3. Principal value of an angle, ratifying an equation.
4. General value of angles.
5. Solution of trigonometric equations.
6. Inverse functions.

VII. Periods, Graphs, Important Limits.

1. Periods of the trigonometric functions.
2. Curves:
   a) sine.
   b) cosine.
   c) tangent.
   d) cotangent.
   e) secant.
   f) cosecant.

D. Solid Geometry

I. Areas of Surfaces.

II. Volumes of Solids.

III. Polyedral Angles.

IV. Spherical Triangles.

E. Surveying

A considerable amount of field work is done with the transit and measuring tape to awaken active interest and to make real the problems of the text.

TIME REQUIRED OF STUDENTS

Five one-hour periods a week throughout the school year are given to the mathematics of the third year. An average of about thirty-five minutes of home work is required. About one-third of the class period is used for study under supervision.

TEXTBOOKS

The best available textbooks are secured for the work of the third year. Slaught and Lennes, *Advanced Course in Algebra*, and Wilczynski, *Plane Trigonometry*, have been used during the last years.

STANDARDS OF ATTAINMENT

In three years the student has accomplished the work usually given in three and one-half years. This is due mainly to the correlation of the various subjects of mathematics, to the reorganization of the material usually taught during the first three years, and to the training obtained through supervised study.

FOURTH-YEAR MATHEMATICS

GENERAL STATEMENT

The course centers around the notion of functionality. It puts the emphasis upon the simple class of cases in which the functional dependence between the quantities involved may be expressed, directly or indirectly, by means of the fundamental operations. The course parallels a similar course in college algebra given in the Junior College of the University.

AIMS AND METHODS

The course is conducted informally with the aim of meeting the needs of a special group of students who intend to continue the study of mathematics or to prepare for the study of the sciences in the college. With their future needs in mind, emphasis is placed upon the fundamental functional relations, the manipulation of formulas, and the graphical interpretation of the functional relations. This graphical treatment forms the introduction to the major topic, namely the elementary discussion of the theory of equations. The minor topics of the course, such as series and determinants, are treated to the extent justified by the ability and the interest of the particular class. The emphasis is less upon formal requirements than upon voluntary contribution. Frequently this takes the form of brief reports by the students to supplement topics previously treated in class. Thus, reports of Forsyth's discussion of the solution of the general equation of the nth degree (*School Science and Mathematics*, December, 1915) supplements the last presentation of Horner's method of finding irrational roots of an equation.

In contrast to the earlier courses a maturer type of scholarship is demanded. Many problems related to the topics treated are

selected from other textbooks. Supervised study is used extensively. Since the student has been trained in habits of study through the supervised study of the preceding courses, he is able to work independently as soon as the theory of the particular lesson has been presented.

CONTENT

I. Review and Further Extension of the Topics of Previous Courses in Algebra.

1. The laws of algebra: commutative, associative, of exponents.
2. The operations in complex problems.
3. Functions.
4. Equations of first, second, and higher degrees in one or more unknowns. Difficult types of simultaneous quadratics. Determinants. Relations between roots and coefficients. Theory of quadratic equations.

II. Theory of Equations.

1. All theorems leading up to, and used in, Horner's method.
2. Logical proofs of these theorems.

III. Series.

1. Binomial theorem for fractional negative exponents.
2. Progressions.
3. Convergent and divergent series.
4. Undetermined coefficients.
5. Permutations and combinations.

TIME REQUIRED OF STUDENTS

Five one-hour class periods a week during one-half of the school year are given to this course. The actual teaching time is approximately one-half of the hour and seldom exceeds that time. The required outside work is limited to thirty-five minutes a day.

TEXTBOOKS AND REFERENCES

The best textbook in college algebra available is used in this course. During the last years Rietz and Crathorne's *College Algebra* was used, but the development paralleled that of Wilczynski's *College Algebra*. The work is supplemented by reports on the elementary discussion of topics found in the journals of secondary-school mathematics.

STANDARDS OF ATTAINMENT

During this course the method of instruction has aimed to develop mathematical power and a mathematical maturity necessary for a successful pursuit of college mathematics. The gap between secondary and college mathematics is thus minimized, the student having learned to appreciate theory of a more difficult type than is ordinarily attempted in the high school.

PREPARATION FOR COLLEGE-ENTRANCE EXAMINATIONS

A course of two periods a week for the first semester and three periods a week for the second semester is required of all. Seniors during certification for admission to colleges which require mathematics in the first year, and of all candidates for recommendation for college-entrance examinations. One-half unit credit is given for this course. Class periods are conducted entirely as supervised study periods, no home work being required. The work of the first two and one-half years is summarized, reviewed, and extended.